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Analysis of Temperature-Dependent Womersley Flow of a Power-Law Fluid through a Porous Medium on Analytic Functions

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Abstract

The goal of this research is to look into the analytic functions for temperature-dependent womersley flow of a power-law fluid through a porous material. Bessel functions are a set of solutions to a second-order differential equation that can appear in a variety of contexts. Mechanics, electrodynamics, elasticity, hydrodynamics, electrical engineering, oscillatory systems, electro engineering, and maritime engineering, heat distribution over an area (smartphones), pressure vessel design, microphone design, solid state physics, and celestial mechanics are some of the applications of Bessel functions. The following tools were used: analytical functions theorems, Microsoft Excel, and Maple Software 2018. The physical properties that describe a power-law fluid passing through a porous media have a significant impact on the flow system's mass transfer. The temperature-dependent parameter should be maintained well enough for optimal production, according to field and production engineers.

Nomenclature and units

1.0 Introduction

Many issues in physics, engineering, and other fields lead to Bessel equations, both linear and nonlinear. The Bessel functions have been known since the 18th century, when mathematicians and physicists began to use differential equations to describe physical phenomena. The same partial differential equations are satisfied by a variety of procedures. The Laplace, dAlembert (wave), Poisson, Helmholtz, and heat (diffusion) equations were named after them. Bessel employed a variety of ways to study these equations (1824).Many technical and biological disciplines deal with transport phenomena in porous media. Chemical engineering uses convective flow through a porous material, particularly in filtration and purification operations. It is used in petroleum technology to investigate the flow of natural gas, oil, and water through oil channels and reservoirs [Batiha et al.,2008; Berenstein & Li, 2005 and Chevalier et al. 2018)

Although many fluids' non-Newtonian behavior has long been understood, the study of rheology is still in its infancy in many ways. As a result, new phenomena are discovered on a regular basis, and new hypotheses on the subject are proposed. More extensive assessments of complex flow and elaborate simulations of the structural and molecular behavior that leads to non-Newtonian phenomena have become possible thanks to advances in computing approaches. Almost every aspect of the economy involves heat transport. In all of these cases, one or more large quantities (such as mass, momentum, and energy, to name a few) are transferred across solid or fluid phases that share a porous medium domain (Hayat *et al.,* 2014).

Pulsatile flow, also known as Womersley flow, is a type of fluid flow that has periodic variations. John R. Womersley (1907–1958) was the first to devise the flow profile. Womersley flow, he claimed, may be applied to blood flow in arteries, electrical circuits, engines, and hydraulic systems (the circulatory system of chordate animals) (as a result of rotating mechanisms pumping fluid).A linear homogeneous differential equation with full solutions given by Bessel and Laguerre polynomials has been the subject of several investigations. However, most of these studies don't go into great detail about analytic functions for temperature-dependent Womersley flow and power-law fluid through a porous media.

Bessel function

Let $$L_m(\beta,t)=\sum_{k=0}^{m}\binom{m+\beta}{m-k}\frac{(-t)^k}{k!} \quad (1)$$

Equation (1) is known as the Laguerre polynomials and is given as:

$$L_m(\beta,t)=\frac{1}{2\sqrt{\pi}}\,e^{t/2}\,(-t)^{-\beta/2-1/4}\,m^{-\beta/2-1/4}\,e^{2\sqrt{-m\,t}}\left(1+\mathrm{O}\left(m^{-1/2}\right)\right)\;(m\to\infty) \quad (2)$$

Equation (2) holds for t in the complex plane, semi-axis; and for $|t|\le r,$ it yield

$$|L_m(\beta,t)|\le m^{-\beta/2-1/4}\,e^{r/2}\,r^{-\beta/2-1/4}\,e^{2\sqrt{m\,r}} \quad (3)$$

which is true in the complex plane for every $m>0$.

Corollary 1

Let $\dfrac{L_m(\beta,t)}{m^{\beta/2}}=e^{t/2}t^{-\beta/2}J_\beta\left(2\sqrt{mt}\right)+\mathrm{O}\left(m^{-6/8}\right)$

be a Bessel function of order m as m approaches infinity and holds on a compact disk of Bessel function as t approaches $(0,+\infty)$, where J_β is the Bessel parameter of order m given below as:

$$J_\beta\left(2\sqrt{m\,t}\right)=\frac{2}{\sqrt{\pi}\;\Gamma\left(\beta+1/2\right)}\left(\sqrt{mt}\right)^\beta\int_0^{\pi/2}\cos\left(\sqrt{m\,t}\,\cos x\right)\sin^{2\beta}x\,dx \quad (4)$$

Equations (3)-(4) yield

$$|L_m(\beta,t)|\le\frac{\pi\,m^\beta}{\sqrt{\pi}\,\Gamma(\beta+1/2)}e^{t/2}\le\frac{\pi\,m^\beta}{\sqrt{\pi}\Gamma(\beta+1/2)}e^{r/2}\le m^{\beta/2-1/4}\,e^{r/2}\,r^{\beta/2-1/4}\,e^{2\sqrt{\pi r}} \quad (5)$$

as $|t|\le r$ whenever $m>0$ holds and is given below as :

$$|L_m(\beta,t)|\le m^{\beta/2-1/4}\,e^{r/2}\,r^{\beta/2-1/4}\,e^{2\sqrt{\pi r}} \quad (6)$$

where

$$\mathrm{M}(r,L_m)=\max\,\max_{|t|\le r}|L_m(\beta,t)| \quad (7)$$

Corollary 2

Bernstein function. Let $\rho(z)$ be an arbitrary π_m satisfying the condition$|\rho(z)|\le 1$, where z is complex, and$|z|\le 1$, then$|\rho'(z)|\le m$, and $|z|\le 1$.Let $\rho(z)$ be a π_m satisfying the condition $|\rho(z)|\le 1$ in $-1\le z\le +1$.Then,

$$\left|\rho'(z)\right| \leq (1-z^2)_m^{-\frac{1}{2}}. \quad (8)$$

Function of Weierstrass With a pre-determined accuracy, polynomials can approximate a continuous function in a finite closed interval. Trigonometric polynomials can be used to approximate a continuous, periodic function of a real variable.

Let $\omega(\delta)$ be the modulus of continuity of given function $f(x)$ continuos in the finite interval$[a,b]$, where

$$\omega(\delta) = max|f(x)' - f(x'')|, \text{ if} |x' - x''| \leq \delta. \quad (9)$$

Then for each m we can find the polynomial $p(x)$ of degree$m$, such that the given interval of length l we have $|f(x)-p(x)| < A\omega({l}/{m})$ where A is an absolute constant.

Let $f(x)$ have a continuous derivative of order μ in the finite interval$[a,b], \mu \geq 1$, and let $\omega_\mu(\delta)$ be the modulus of continuity $f^{\mu}(x)$. Then, a polynomial $p(x)$ of degree $m+\mu$ exits, such that

$$|f(x)-p(x)| < C\left({l}/{m}\right)^{\mu}\omega_\mu({l}/{m}),$$

$$\left|f'(x)-q(x)\right| < C\left({l}/{m}\right)^{\mu-1}\omega_\mu({l}/{m}) \quad (10)$$

$$\left|f'(x)-q(x)\right| < K\omega_\mu\left({l}/{m}\right)$$

where$C$ is a constant depending only on μ, $l=b-a, q(x)$ is a proper $\pi_{m+\mu-1}.f(x)-\int_a^x q(t)\,dt$, satisfyies Lipschitz condition with $\lambda = k\left({l}/{m}\right)^{\mu-1}\omega_\mu({l}/{m})$,

It yields π_m, say$\sigma^x, \sigma^{(x)}$, such that

$$\left|f(x)-\int_a^x q(x)\,dt-\sigma(x)\right| < K'({l}/{m})^{\mu}\omega_\mu\left({l}/{m}\right), \quad (11)$$

$$\left|\sigma'(x)\right| < K''\left({l}/{m}\right)^{\mu-1}\omega_\mu\left({l}/{m}\right)$$

Generalization of growth parameter.$P_n^{\alpha,\beta}(x)$is an extension to arbitrary complex values of the parameters$\alpha and \delta$. It is a polynomial in $x, \alpha and \beta$ and π_n is denoted by $P_n^{(\alpha,\delta)}(x)$

$\binom{n}{l}P_n^{(l,\delta)}(x) = \binom{n+\delta}{l}\left(\frac{x-1}{2}\right)^l P_{n-l}^{(l,\delta)}(x)$,where$l$ is an integer

$1 \leq l \leq n$, and

$$\binom{n}{k-1}P_{n-l}^{(l,\delta)}(x) = \binom{n+\alpha}{n-k+1}P_{k-1}^{(\alpha,\delta)}(x), \quad (12)$$

$n+\alpha+\delta+k=0$, where k is an integer, $1 \leq k \leq n$

Laguerre Polynomials

Given$\{L_n^{(\alpha)}(x)\}$, for $\alpha > -1\, y$ define by the condition of orthogonality and normalization as follows:

$\int_0^\infty e^{-\alpha}x^\alpha L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{nm}$, where $n,m = 0,1,2\ldots$, coefficient of x^n in the polynomial $L_n^\alpha(x)$ of degree $n$have the sign (-1).

$$L_n^{(0)}(x) = L_n(x)$$

The temperature-dependent power-law fluid of Womersley flow through a porous material is studied using analytic functions in this study. As a result, this research will have a favorable theoretical impact on the development of modeling strategies and procedures for temperature-dependent factors. Kumar (2017) looked at the generalized growth of analytic function solutions to second-order linear homogeneous partial differential equations. In a convergent series of laguerre polynomials, he shows the coefficient characterizations of generalized order and generalized type of the solution. Variable permeability dependent on temperature in the presence of an Arrhenius reaction, flow through a porous medium were taken into account by Peter *et al.*,(2019). The model's existence and distinctiveness were established in his work.

In some Banach spaces, Vakarchuk and Zhir (2015) investigated the optimal polynomial approximations of full transcendental functions of many complex variables. The limiting relationships between the stated growth features were discovered. In a finite disk, Kumar and Basu (2014) studied the growth and $L\delta-$approximation of Helmholtz equation solutions. The effects of Joule heating and thermal radiation on the flow of third-grade fluid across a radiative surface were investigated by Hayat et al. (2014). Zhang and Hu (2012) looked at a linear homogeneous partial differential equation with laguerre polynomials representing the full solution. In some Banach spaces, the best polynomial approximation of the full transcendental functions of multiple complex variables. A linear homogeneous partial differential equation with full solutions was studied by Wang *et al.*,(2012).

Vakarchuk and Zhir (2015) studied the best polynomial approximations of entire transcendental functions of many complex variables in some Banach spaces. The limiting relations between the indicated characteristics of growth were obtained. Kumar and Basu (2014) considered growth and $L\delta-$approximation of solutions of the Helmholtz equation in a finite disk. Hayat et al., (2014) examined the effect of Joule heating and thermal radiation in flow of third grade fluid over radiative surface. Zhang and Hu (2012) investigated on a linear homogeneous partial differential equation with entire solutions represented by laguerre polynomials. On the best polynomials approximation of entire transcendental functions of many complex variables in some Banach spaces. Wang *et al.*, (2012) examined a linear homogeneous partial differential equation with entire solutions represented by laguerre polynomials. Hu and Yang (2010) looked at a linear homogeneous partial differential equation with Bessel polynomials as the full solution. Hu and Yang (2009) investigated global solutions of second-order homogeneous linear partial differential equations as follows:

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + (2t+2)\frac{\partial u}{\partial t} - 2z\frac{\partial u}{\partial z} = 0$$

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} - z\frac{\partial u}{\partial z} + t^2 u = 0 \qquad (13)$$

These solutions are closely related to Bessel functions and Bessel polynomials, according to the findings $(t,z) \in C^2$. As a result, the goal of this research is to fill in the gaps in the literature by using analytic functions to include the temperature-dependent power-law fluid of Womersley flow through a porous media.

2.0 Materials and Methods

Mathematical Formulation

The governing equations with the initial circumstances assume the following forms, according to (Wang *et al.*, 2012 and Kumar, 2017). For the numerical solution, the researchers used Bessel equations, Microsoft Excel, and Maple Software 2018.

$$\delta\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{dp}{dz} + \frac{\omega}{\rho}\frac{\partial}{\partial z}\left(\beta(T)\left(\frac{\partial u}{\partial z}\right)\right) + z^n \upsilon\left(\frac{\partial u}{\partial z}\right) - \frac{\mu}{\rho K}u = 0 \qquad (14)$$

$$u(z,0) = 0 \qquad (15)$$

$$u_t(z,0) = 1 \qquad (16)$$

where equation(14) is called Bessel equation of order $\beta, \beta \epsilon \mathbb{C}$, $\beta(T)$ is a true variable (non-negative and temperature-dependent), $\upsilon = \frac{\mu_0}{\rho}$, $\beta(T) = \beta_0 e^{-b(T-T_0)}$, $u = f(z,t)$ (t being the time), $T = \theta(T - T_0) + T_0$, μ_0 is a metric that measures the constancy of a flow, $z, t \in C^2$, $\omega, \lambda, \upsilon$ this is a true set of positive characteristics, n is the flow behavioural index, ρ is the density of the fluid, ω is the fluid's angular frequency, $n < 1$ (pseudo-plastic fluids), z being the distance from the symmetry axis, μ is the dynamic viscosity, $u(z,t)$ is the dimensional velocity field along the z axis, K is the permeability, p is the pressure, T is the temperature and b is the exponential variation.

Theorem

Problem (14**)** has a unique solution $u = w(t,z)$ on $\mathbb{C}^2$ only if, and only if, the boundary requirements are met $u = w(t,z)$ on the collection of complex numbers, it has a power series expansion.

$$w(t,z) = \sum_{k=0}^{\infty} a_k L_K(\lambda,t) z^k \qquad (17)$$

such that

$$limsup \sqrt[k]{|a_k|} = 0 \qquad (18)$$

Assuming that $w(t,z)$is a unique solution on complex plane$\mathbb{C}^2$,

let $\underset{|t\le r| \quad ,|z|\le r}{Max} |u(t,z)|$

$$O(0,u) = \lim_{r\to\infty} sup \frac{(ln^+)^2 M(r,u)}{lnr} \qquad (19)$$

where the limit of $In^+ x$ exists

$$In^+ x = \begin{cases} In^+, & if x \ge 1 \\ 0, & if x < 1 \end{cases} \qquad (20)$$

Proof:

Following Szego (1975) the properties of Laguerre polynomials $L_k(\delta,t)$ are given

$$L_k(\delta,t) = \frac{2}{4\sqrt{\pi}} e^{t/4} (-t)^{-\delta/4 - 1/8} k^{\delta/4 - 1/8} e^{4\sqrt{-kt}} (1 + 0(k^{-1/4})) \qquad (21)$$

as$k \to \infty$ satisfies fot t in complex domain and for $|t| \le r$ equation (21) yields

$$|L_k(\delta,t)| \le k^{\frac{\delta}{4}-\frac{1}{8}} e^{r/4} r^{-\frac{\delta}{4}-\frac{1}{8}} e^{4\sqrt{kr}}$$

$$|L_k(\delta,t)| \le k^{\delta/4 - 1/4} e^{r/4} r^{-8/4 - 1/8} e^{4\sqrt{+kr}} \qquad (22)$$

Equation (22) holds as n approaches infinity.

Let $u = w(t,z)$ be a unique solution on $\mathbb{C}^2$ satisfying problem (14) we obtain the power series expansion as follows:

$$w(t,z) = \sum_{k-0}^{\infty} \frac{f_k(t)}{k!} z^k \qquad (23)$$

where

$$f_k(t) = \frac{\partial^k w}{\partial z^k}(t,0) \tag{24}$$

Equations(14)-(24) become

$$\delta\frac{df}{dt} + z^n \nu_k \frac{df}{dz} + \beta_1(1+\gamma\theta)\frac{d^2 f}{dz^2} + c - \frac{f}{Da} = 0 \tag{25}$$

Equation (25) is a solution of (14).

$$y_k(\delta,t) = RL_k(\delta,t)\ln t + \sum_{i=0}^{\infty} c_i t^i$$

where$R \neq 0$ and c_i are constants.

$$f(t) = k!\, a_k L_k(\delta,t) + b_k y_k(\delta,t) \tag{26}$$

For a regular singular point of equation (14) to be an analytic function $t = 0$, then equation (26) yields

$$w(t,z) = \sum_{k=0}^{\infty} a_k L_k(\delta,t) z^k,\; f(0,z) = \sum_{k=0}^{\infty} a_k L_k(\delta,0) z^k$$

which is a function of the entire problem

Then,

$$\lim_{k\to\infty} sup \sqrt[k]{|a_k| L_k(w,0)} = 0$$

$$\therefore \delta\frac{\partial w}{\partial t} + c + z^n v\left(\frac{\partial w}{\partial z}\right) + \frac{w}{Da} - \beta_1(1+\gamma\theta)\frac{\partial^2 w}{\partial z^2}$$

$$= \sum_{k=0}^{\infty} a_k\left[\delta\frac{dL_k(\delta,t)}{dt} + c + k^n v L_k(\delta,t) - \beta_1(1+\gamma\theta)\frac{d^2 L_k(\delta,t)}{dt^2}\right] z^k = 0 \tag{27}$$

where $\beta_1 = \dfrac{\omega\beta_0}{\rho}$ and equation (14) holds for all $(t,z) \in \mathbb{C}^2$

Theorem 1: Existence and uniqueness of linear differential equation systems' solutions

We consider the initial value system

$$x_1^1 = f_1(t, x_1, x_2, ...x_n), \qquad x_1(t_0) = x_{10}$$

$$x_2^1 = f_2(t, x_1, x_2, ...x_n), \qquad x_2(t_0) = x_{20}$$

$$.$$

$$. \tag{28}$$

$$x_n^1 = f_n(t, x_1, x_2, ...x_n), \qquad x_n(t_0) = x_{n0}$$

$$x = \begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}, \quad f(t,x) = \begin{bmatrix} f_1(t, x_1, x_2, ...x_n) \\ . \\ . \\ f_n(t, x_1, x_2, ...x_n) \end{bmatrix},$$

$$x_0 = \begin{bmatrix} x_{10} \\ . \\ . \\ x_{n0} \end{bmatrix}$$

We then write it in compact form

$$x^1 = f(t,x), \quad x(t_0) = x_0 \tag{29}$$

Theorem 2 (Derrick and Grossman, 1976)

Let D denote the region [in $(n+1)-$ dimensional space, one dimension for t and n dimensions for the vector x]

$$|t - t_0| \le a, \; \| \; x - x_0\| \le b, \tag{30}$$

and suppose that $f(t,x)$ satisfies the Lipschitz condition

$$\| f(t,x_1) - f(t,x_2)\| \le \; K_i \; \|x_1 - x_2\| \tag{31}$$

whenever the pairs (t,x_1) and (t,x_2) belong to D, where K_i is a positive constant. Then there is a constant $\delta_i > 0$ such that there exists a unique continuous vector solution $x(t)$ of the system of solution (28) in the interval $|t - t_0| \le \delta_i$.

Alternatively, if $\partial f_i / \partial x_j,\, i,\, j = 1,2,........n$ they go on D indefinitely, they're bound together on D, and conditions (30) and (31) are met.

Theorem 3

Let D be the domain of existence of f and θ Let $c, v, k_2, \beta_1 and D_a$ are positive real constants other than zero. Then equation (14) satisfying the boundary condition has a unique solution in the domain D.

For a regular singularity point $t = 0$, equation (14) becomes

$$z^n v \frac{df}{dz} + \beta_1(1+\gamma\theta)\frac{d^2 f}{dz^2} + c - \frac{f}{Da} = 0 \qquad (32)$$

$$\frac{d^2 f}{dz^2} = \frac{1}{\beta_1(1+\gamma\theta)}\left[-z^n v \frac{df}{dz} - c + \frac{f}{Da}\right]$$

Let $x_1 = z,\ x_2 = f,\ x_3 = f',\ x_4 = \theta$

$x_1' = f_1(x_1,\ x_2,\ x_3,\ x_4)$

$$x_2' = f_2(x_1, \quad x_2, \quad x_3, \quad x_4) = x_3$$

$$x_3' = f_3(x_1, \quad x_2, \quad x_3, \quad x_4) = f''$$

$$x_4' = f_4, (x_1, \quad x_2, \quad x_3, \quad x_4)$$

Satisfying

$$0 \le x_1 \le \infty$$

$$-k_1 \le x_2 \le k_1$$

$$-\alpha_1 \le x_2 \le \alpha_1$$

$$-k_2 \le x_4 \le k_2 \qquad (33)$$

Then,

$$\left|\frac{\partial f_2}{\partial x_1}\right| \le 0, \left|\frac{\partial f_2}{\partial x_2}\right| \le 0, \left|\frac{\partial f_2}{\partial x_3}\right| \le 0, \left|\frac{\partial f_2}{\partial x_4}\right| \le 0 \qquad (34)$$

$$\frac{\partial f_3}{\partial x_1} \le \left|\frac{1}{\beta_1\gamma(1+\gamma k_2)}\upsilon k_1\right| \qquad (35)$$

$$\left|\frac{\partial f_3}{\partial x_2}\right| \le \left|\frac{1}{\beta_1 Da(1+\gamma k_2)}\right| \qquad (36)$$

$$\frac{\partial f_3}{\partial x_3} \le \left|\upsilon x_1\left(\frac{1}{\beta_1\gamma(1+\gamma x_4)}\right)\right| \le 0$$

$$\frac{\partial f_3}{\partial x_4} \le \left|\frac{1}{\beta_1\gamma}\left(\upsilon x_1 x_3 + c + \frac{x_3}{Da}\right)\right| \le \left|\frac{1}{\beta_1\gamma}\left(c + \frac{k_1}{Da}\right)\right|$$

$$k\ (\max) = \left|\frac{1}{\beta_1\gamma}\left(c + \frac{k_1}{Da}\right)\right|$$

$$\therefore k = \frac{1}{\beta_1\gamma}\left(\upsilon\alpha_1^{n-1} + c + \frac{k_1}{Da}\right) \qquad (37)$$

$k < \infty$, since k, c, β_1, γ and Da are positive real constants other than zero. Therefore, k exists. Hence, $\frac{\partial f_i}{\partial x_j}, i, j = 1,2,3$ are Lipschitz continuous and are bounded in D for each bounded x_1 and x_2. As a result, the problem (37) has a unique solution and it is well posed. This completes the proof.

3.0 Results

From Equation (30) let $n = -1,\ \theta = \theta_0$ and c is negligible equation (30) becomes

$$z^n v \frac{df}{dz} + \beta_1(1+\gamma\theta_0)\frac{d^2 f}{dz^2} - \frac{f}{Da} = 0 \qquad (38)$$

$$f(z) = z^2 \sum_{n=0}^{\infty} a_n z^n \qquad (39)$$

Equations (38)-(39) yield

$$z^n b \sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} + \upsilon z^{n+2}\sum_{n=0}^{\infty} n a_n z^{n-1} - \frac{z^2}{Da}\sum_{n=0}^{\infty} n a_n z^{n-1} = 0$$

$$b\sum_{n=0}^{\infty} n(n-1)a_n z^n + \upsilon z \sum_{n=0}^{\infty} n a_n z^{n-2} - \frac{1}{Da}\sum_{n=0}^{\infty} n a_n z^{n+2} = 0 \qquad (40)$$

Let $n + 2 = m \Rightarrow n = m - 2$

$$b\sum_{m=0}^{\infty} m(m-1)a_m z^m + \upsilon\sum_{m=1}^{\infty} m a_m z^m - \frac{1}{Da}\sum_{m=2}^{\infty} a_m z^m = 0 \qquad (41)$$

The first term is equal to zero for $m = 0,\ m = 1$, in the first sum and m equal to zero in the second sum when $m \ge 2$, we obtain

$$bm(m-1)a_m + vma_m - \frac{a_m}{Da} = 0 \qquad (42)$$

Let $v = b$, equation (42) becomes

$$vm(m-1)a_m + vma_m - \frac{a_m}{Da} = 0 \tag{43}$$

$$vm^2 a_m - vma_m + vma_m - \frac{a_{m+2}}{Da} = 0 \tag{44}$$

$$vm^2 a_m - \frac{a_{m+2}}{Da} = 0 \tag{45}$$

$$a_m = \frac{a_{m+2}}{vm^2 Da}$$

$$a_1 = 0$$

$$a_m = \frac{a_0}{vz^2 Da} = \frac{a_0}{z^2 Da} \tag{46}$$

$$a_2 = \frac{a_0}{vz^2 Da} = \frac{a_0}{z^2 \beta_1 (1+\gamma_1\theta_0) Da})$$

$$a_4 = \frac{a_2}{z^2 4^2 \beta_1 (1+\gamma_1\theta_0) Da}, \ldots$$

$$a_{2m} = \frac{(-1)^m}{2^2.4^2....(2m)^2} = \frac{(-1)^m a_0}{2^{2m}(m!)^2 \beta_1 (1+\gamma_1\theta_0) Da}$$

$$f(z) = a_0 + \frac{a_0}{2^2 \beta_1 (1+\gamma_1\theta_0) Da} z^2 + \frac{a_0}{2^2 4^2 \beta_1 (1+\gamma_1\theta_0) Da} z^4 + \ldots$$

$$= \frac{a_0}{\beta_1 (1+\gamma_1\theta_0) Da} \sum_{n=2}^{\infty} (-1)^n \frac{\left(t/2\right)^2}{(n!)^2} \tag{47}$$

For $a_0 = 1$, the Bessel function of order zero is defined by the preceding equation.

It belongs to the first category and is identified by $J_\Lambda(z)$, regardless of the parameter approaches

$$J_\Lambda(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\Lambda\pi}{2} - \frac{\pi}{4}\right) \tag{48}$$

Asymptotically as t approaches infinity $J_0(z)$ approaches zero and $J_0(z)$ contains an endless number of zeros, which is more similar to $\cos\left(z - \frac{\pi}{4}\right)$ for large values of z. As a result, equation (49) is formed.

$$J_\Lambda(z) = \frac{1}{\beta_1 (1+\gamma_1\theta_0) Da} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) \tag{49}$$

Using F solve in Maple Software 2018 equation (49) yields

$$J_\Lambda(z) = \frac{\sqrt{2}\cos\left(z - \frac{\pi}{4}\right)}{\pi z \beta_1 (1+\gamma_1\theta_0) Da} \tag{50}$$

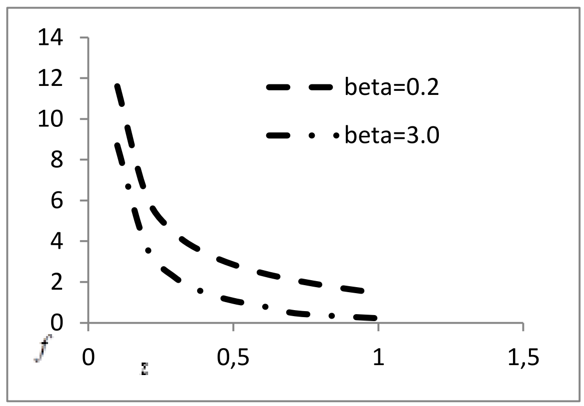

Figure 1: Graph of the velocity function f for various values of temperature -dependent parameter β_1 and Darcy number, $Da = 0.1$.

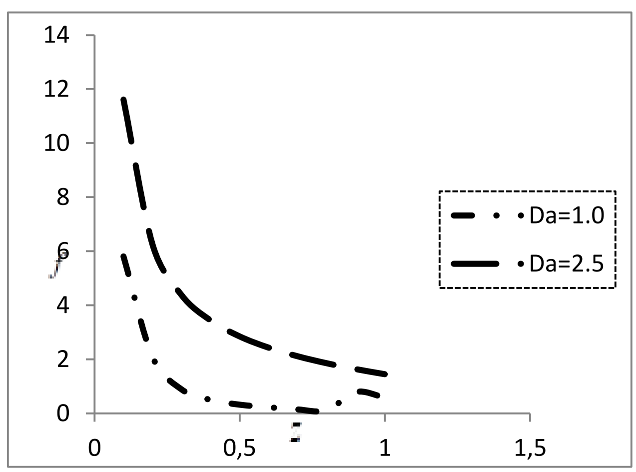

Figure 2: Graph of the velocity function f for various values of Darcy number, and temperature -dependent parameter $\beta_1 = 0.3$.

4.0 Discussions

The flow of a Womersley power-law fluid through a porous media was investigated, as well as the dimensionless boundary conditions, which were then numerically solved using an algorithm in the symbolic computer algebra program Maple 18 software. Figures 1 and 2 depict the findings of this investigation graphically. The effect of temperature-dependent parameter, Darcy number, and power-law index on the flow of Womersley power-law fluid through a porous material is revealed in this study.When a Pseudoplastic fluid has temperature-dependent parameters and a power-law index, the flow decreases as the temperature-dependent parameter increases.The Darcy number is useful in determining the physical principles that control power-law fluid flow in porous media. With increasing porosity and power indexes, the temperature-dependent parameter rises.

5.0 Conclusions

The following conclusions can be drawn from the study's findings: With each rise in a temperature-dependent parameter in the flow system, the energy transfer increases, reducing the mass flow in the system. The physical properties that describe a power-law fluid passing through a porous media have a significant impact on the flow system's mass transfer. Due to space and time constraints, the study of Womersley flow of a power-law fluid through a porous medium is inexhaustible in a single work. The temperature-dependent parameter should be maintained well enough for optimal production, according to field and production engineers. Parts of the findings are also expected to serve as the foundation for more mathematical modeling and study of Bessel's equations in the future.

The outcome of this research demonstrates a clear and improved understanding of Womersley flow of a power-law fluid through a porous material, with the following specific contributions to knowledge:

1. Analytic functions are used to study the temperature-dependent Womersley flow of a power-law fluid through a porous media.
2. The existence and uniqueness of model solutions were specified as criterion.
3. The model's impacts of the linked parameter are calculated.
4. It is noticed that as the temperature-dependent parameter is increased, the fluid velocity increases steadily.

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Declaration of conflict of interest

The authors state that the publishing of this paper is free of any conflicts of interest.

References

Batiha, B. M., Noorani, S.M. and Hashim, I. (2008). Numerical simulations of systems of PDEs by variational iteration method, Phys. Lett. A, 822–829

Berenstein, C.A. and Li, B.Q. (2005) "On certain first-order partial differential equations in Cn," in *Harmonic Analysis, Signal Processing and Complexity*, Birkhauser, Boston, MA, USA.vol. 238 of *Progr. Math.*, pp. 29–36

Chevalier, T. Chevalier, T., Clain, C. X., Dupla, X., Canou, J.C., Rodts, J. S. and Coussot, P. (2018).Darcy's law for yield stress fluid flowing through a porous medium. *Journals of Non- Newtonian Fluid Mechanics*, Elsevier, 195, pp.57-66,

Hayat, T., Anum, S and Ahmed, A (2014). Effect of Joule heating and thermal radiation in flow of third grade fluid over radiative surface. Nonlinear Analysis and Applied Mathematics Research Group, Faculty of Science, Saud Arabia,vol.9,issue1,pp1-12. http:/www.plosone.org. Accessesed March, 2014.

Hidetoshi (2007). Coupling of two Partial Differential Equations and its application. Journals of *Publ. RIMS, Kyoto Univ.*43 (2007), page 535–583.

Hillen, T., Diana White, D., De-Vries, G. and Adriana Dawes (2017) "Existence and uniqueness for a coupled PDE model for motor-induced microtubule organization" *Journal of Biological Dynamics*, 11:sup2, 294-315.

Hu, P.C. and Yang, C.C. (2009). Global solutions of homogeneous linear partial differential equations of the second order. *Michigan Mathematical Journal*, vol. 58, no. 3, pp. 807–831.

Hu, P.C. (2008). Introduction of Function of One Complex Variable. *Science Press*, Beijing, China

Hu, P.C. and Yang, C.C. (2010). Malmquist type theorem and factorization of meromorphic solutions of partial differential equations. *Journal ofComplex Variables. Theory and Application. An International Journal*, vol. 27, no. 3, pp. 269–285.

Hu, P.C. and Li, B.Q. (2011). On meromorphic solutions of nonlinear partial equations of first order. *Journal of Mathematical Analysis and Applications*, vol. 377, no. 2, pp. 881–888.

Iyiola, F.P. (2017). On steady flow of a reactive viscous fluid in a porous cylindrical pipe. *Open Journal of Fluid Dynamics*,7, 359-370, 2017.

James, S.A. (2008). Womersley flow. Accessed may, 2018

Kapoor, G.P. and Nautiyal, A. (1988) "Growth and approximation of generalized bi-axially symmetric potentials," *Indian Journal of Pure and Applied Mathematics*, vol. 19, no. 5, pp. 464–476.

Khan, K., Akbar, M.A., and Koppelaar, H. (2015). Study of coupled nonlinear partial differential equations for finding exact analytical solution *R. Soc Open Sci* 140406. PMC4632569

Kumar, D. and Basu, A. (2014) "Growth and L□ -approximation of solutions of the Helmholtz equation in a finite disk," *Journal of Applied Analysis*, vol. 20, no. 2, pp. 119–128.

Kumar, B. S. and Kumar, P. (2017). Fractional variational iteration method for solving fractional Partial Differential Equations with proportional delay. *International Journal of Differential Equations*, Article ID 5206380

Li, B.Q. and Saleeby, E.G. (2003). Entire solutions of first-order partial differential equations, *Complex Variables. Theory and Application. An International Journal*, vol. 48, no. 8, pp. 657–661,

Li, B.Q. (2005) "Entire solutions of certain partial differential equations and factorization of partial derivatives," *Transactions of the differential American Mathematical Society*, vol. 357, no. 8, pp. 3169–3177

McCoy, P.A. (1986). Approximation of pseudo analytic functions on the unit disk. *Complex Variables. Theory and Application. An International Journal*, vol. 6, no. 2-4, pp. 123–133, 1986

Miadonye, A. and Amadu M. (2014). Analytical derivation of the temperature dependence of absolute permeability of a porous medium.*Journal of Petroleum Science Research (JPSR*) Volume 3 Issue 4, October 2014.

Peter, B.A., Ogunsola, A.W., Itodo, A.E., Idowu, S.A. and Mundu, M.M. (2019). Reacting flow of temperature-dependent variable permeability through a porous medium in the presence of Arrhenius reaction. *Amer. J. Mathem. and Comp. Sci.*4(1):11-18.

Rawashdeh, M.S. and Maitama, S. (2014). Solving coupled system of nonlinear PDEs using the natural decompositi on method,*International Journal of Pure and Applied Mathematics* Volume 92 No. 5, 757-776.

Srivastava, G.S. and Kumar, S. (2009). On approximation and generalized type of entire functions of several complex variables. *European Journal of Pure and Applied Mathematics*, vol. 2, no. 4, pp. 520–531.

Tol, H.T., de-Visser, C.C., and Kotsonis, M. (2016). Model reduction of parabolic PDEs using multivariate splines, *International Journal of Control*.

Vakarchuk, S.B. and Zhir, S.I. (2008). On the best polynomial approximation of entire transcendental functions of generalized order. *Ukrainian Mathematical Journal*, vol. 60, no. 8, pp. 1183–1199.

Vakarchuk, S. B. and Zhir, S.I. (2015). On the best polynomial approximations of entire transcendental functions of many complex variables in some Banach spaces. *Ukrainian Mathematical Journal*, vol. 66, no. 12, pp. 1793–1811.

Vajravelu, K., Harmand, S. and Roper, T. (2012). Unsteady flow and heat transfer in a thin film of Ostwald–de-Wall liquid over a stretching surface. *Journal of Comu Nonlinear Science Numer Simulat*. 17: 4163-4173.

Wang, X.L., Zhang, F.L. and Hu, P.C. (2012). A linear homogeneous partial differential equation with entire solutions represented by Laguerre polynomials. *Abstract and Applied Analysis*, vol. 2012, Article ID 609.

Wazwaz,A.M. (2007.) The variational iteration method for solving linear and nonlinear systems of PDEs, *Comput. Math. Appl*. 895–902.