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An Implicit Steger-Warming Flux Vector Splitting Method for the Transient Two-Phase Flow in a Pipe

[1]Yale, I.D, and [2]Norsarahaida Amin

[1]Department of Electrical, Telecommunication and Computer Engineering, Kampala International University. Uganda.

[2]Department of Mathematical Science, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor, Johor Bahru, Malaysia

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Abstract

This paper presents a one-dimensional single pressure model representing the system of the two-phase flow for predicting and modelling the flow physics of transient two-phase flow. This model consists the interfacial interactions properties of the fluids at the interface and also with walls of the pipe. The governing equations were solved numerically using the implicit Steger-Warming flux vector splitting method. Numerical results on air-water compressible flow problems are preformed and analyzed. A numerical computation for test case problem separation was evaluated. The results for the liquid fraction and velocity for the separation case was presented and shows good agreement with analytical solution. The discretization for various cells and time evolution are presented that give an insight for the convergence and stability of the numerical scheme for on the test case.

Nomenclature and units

ρ_k	Density of the fluids
α_k	Volume fraction of phases
V_k	Velocity of phases
k	Phases
G	Gas phase
L	Liquid phase
τ_k	Wall momentum of phases
τ_i	Interfacial exchange term
θ	Angle of inclination
P	Pressure term
P_c	Pressure correction term
t	Time
x	Direction of flow
g	Acceleration due to gravity

1.0 Introduction

The two-phase gas-liquid two-phase flow is considered to be the most important aspect of multiphase flow for its widely application in industries, such as in the production and transportation of oil and gas in the petroleum industry. It also plays a role in evaporation, boilers, condensers, submerged combustion systems, sewerage treatment plant, air condition and refrigeration plant, meteorology and other natural phenomena.

In many flow processes of interest for engineering, environmental, and biological systems are mostly considered to be two-phase flow in nature or include at least some features of two-phase flow. This is why enormous attention has been drawn to two-phase flow during the last decades. Interest on two-phase flow has been keen, though, mathematical modelling on the two-phase flow are relatively rare and are mostly limited to single phase gas or the homogeneous two-phase flow model then later dealing with more complex two-phase flow under heterogeneous and non-equilibrium conditions.

Recently, the numerical simulation of the wave propagation processes on two-phase flow is based on a newly developed hyperbolic two-fluid model which seems to allow an algebraic of the complete eigenspace (eigenvalues and related eigenvectors) (Stadtke, 2006). A two phase flow based on the eight equation model of two-pressure model was developed by Ransom and Hicks (1988) a model used by Baer and Nunziato (1986) in their work. A modification of the model was proposed by Drew and Passman (1999) and Gonthier and Powers (2000). In their work, the model consisting of seven partial differential equations: one representing the transport equation for the volume of fraction, two describing the mass of each fluid, two for the momentum of each fluid and two for the energy of each fluid. An approach for predicting two fluid model of two phase flow phenomenon for the determination of computational efficiency in two phase modelling was investigated (Coquel *et al.*, 1997). The method of upwind scheme based on finite volume method was adopted for the numerical solution. Saurel and Abgrall (1999) proposed a model and solution method for two phase compressible flows for mixture and multifluids flows. The multiphase Godunov method were used to solve the system at each mesh point and also simulate interfacial problems between pure fluids and multiphase mixtures for several test cases where fluids have compressible behaviour as well as incompressible phases. A simulation of multifluids compressible flows using a simple second order and fully Eulerian numerical method governed by the Stiffened gas equation of state was presented by Saurel and Abgrall (1999). The method was used to compute a strong shock wave propagating in a liquid with a gas cylinder. Multiphase flow modelling using homogeneous equilibrium model (HEM) and Riemann solvers was also studied by Shyue Keh-Ming (1999). In their work, a numerical computational for air and water flow using a hybrid equation of state based on a combination of perfect gas law to model the gas phase and Stiffened equation of state or van der Waals equation of state to model the liquid phase. The numerical resolution of multicomponent problems with a Van der Waals fluid was extended to a more general case with real material characteristics by a Mie Gruniesen equation of state Shyue (2001). A simulation of one dimensional two-phase flow two-fluid model in pipelines where pressure relaxation term was added (Loillier *et al.*, 2005). The systems of equations consist of five-time dependent partial differential equations solved explicitly by a finite volume approach based on the AUSMDV. A numerical computation on air-water flow problems were performed an analysed. Gessner and Barbosa (2009) carried out a numerical study for solving two-phase homogeneous transient flows based on the split coefficient matrix (SCM). The numerical modelling for slug initiation and growth in horizontal ducts using two-fluid model of Pressure Free Model (PFM) (Ansari & Shokri, 2011). The transient two-fluid equations were solved numerically by a class of high resolution slug capturing methods.

Steger and Warming (1981) developed a method for splitting the system of equation into component of the same characteristics behaviour for a one-dimensional inviscid equation of gasdynamics. The explicit and implicit numerical algorithms were devised and tested for the split system of equations. A transient two-phase flow homogeneous equilibrium model solved using the splitting method of one sided spatial difference operator of finite difference equation (Liou & Steffen, 1991). A numerical approximation of two phase models of non-equilibrium two-phase flows of six balance equations using the splitting scheme for the system of equations was presented by Rascle and EL Amine (1997). The kinetic upwind scheme was considered, the algorithm were used to compute the flow regimes evolving from mixture to single phase flows and vice versa. A numerical test such as phase separation and phase transition were performed. A compressible transient two phase flow model for the simulation of pipeline flow (Daniels *et al.*, 2002). The numerical computation for the transient flow for both vertical and horizontal two phase flows on air and water sedimentation flows and a two phase shock tube problem using the method of implicit solvers. The Steger-Warming flux vector splitting approach and the NND scheme for studying the hyperbolic partial differential equation of Euler equation (Xinfeng *et al.*, 2013). The gas-liquid-solid three phase mixed flow for the coupling hydraulic transient problems in pipeline (Chen, *et al.*, 1998). The problem was numerically solved using a finite difference scheme based on the Steger and Warming flux vector splitting. The flux subvectors were discretized by the Lax-Wedroff central difference scheme and the Warming-Beam upwind difference scheme with second-order precession in both time and space.

An idea of two-phase compressible flow is based on the consideration and averaging of balance conservation for each phase. With this, the two-phase flow can be said to be an averaged continuum whereby the interfacial interactions was considered (Drew & Passman, 1998). A single pressure two-fluid model of two-phase flows describe more details the two-phase two-fluid model which consists of the interfacial interactions with an extra differential terms to the governing equations, such as the virtual mass and interfacial pressure forces (Omgba, 2005; Stadtke & Franchello, 2001). The addition of the differential terms modified the system of governing equations of two-phase two-fluid model to give a real eigenvalues and a complete set of linearly eigenvectors and hence makes it hyperbolic.

In this paper, the transient two-phase flow one dimensional single model is considered. This model however, represents the mathematical modelling of the two-phase flow system. The numerical simulation on water faucet and separation test case will be used to evaluate the stability, accuracy and convergences of the numerical solution of implicit Steger Warming flux vector splitting method.

2.0 Mathematical Analysis

The two-fluid model is represented by two sets of conservation equations for the balance of mass, momentum and energy for each of the phases. The one-dimensional form of the model is obtained by integrating (area averaging) the flow properties over the cross-sectional area of the flow. The transfer of momentum and energy between the walls and the fluids are included in the source terms in the equations. Moreover, the dynamic interaction between the phases across the interfaces is modeled using inter-phase forces that appear as source terms in the transport equations.

The present study considered the transport equations for the transient flow of the one-dimensional single pressure two-fluid model of conservation mass and momentum equations for gas and liquid phases. The flow is assumed to be isothermal with no mass and heat transfer conditions. The model description is represented in Figure 1 for the direction of the coordinate axes. Following these axes and the assumptions above, the governing equations is presented as follows;

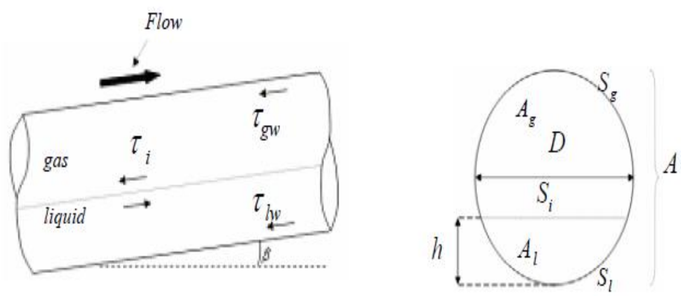

Figure 1 Pipe cross-section and side view of two-phase flow in a pipe with relevant properties.

The mass conservation equation for the gas phase is expressed as:

$$\frac{\partial \rho_G \alpha_G}{\partial t} + \frac{\partial \rho_G \alpha_G V_G}{\partial x} = 0 \quad (1)$$

The mass conservation equation for the liquid phase is;

$$\frac{\partial \rho_L \alpha_L}{\partial t} + \frac{\partial \rho_L \alpha_L V_L}{\partial x} = 0 \quad (2)$$

The momentum conservation equation for the gas phase is;

$$\frac{\partial \rho_G \alpha_G V_G}{\partial t} + \frac{\partial \rho_G \alpha_G V_G^2}{\partial x} = -\alpha_G \frac{\partial P}{\partial x} - \alpha_G \rho_G g \frac{\partial h_l}{\partial x} \cos\theta - \alpha_G \rho_G g \sin\theta - \tau_I S_I - \tau_G S_G \quad (3)$$

The momentum conservation equation for the liquid phase is;

$$\frac{\partial \rho_L \alpha_L V_L}{\partial t} + \frac{\partial \rho_L \alpha_L V_L^2}{\partial x} = -\alpha_L \frac{\partial P}{\partial x} - \alpha_L \rho_L g \frac{\partial h_L}{\partial x} \cos\theta - \alpha_L \rho_L g \sin\theta + \tau_I S_I - \tau_L S_L \quad (4)$$

where ρ_k, α_k and V_k represents the density, volume fraction and velocity of phase k (G is the phase and L is the liquid phase). The variable τ_k and τ_i are the wall momentum and interfacial exchange terms, θ is the angle of inclination to the horizontal and P denotes the pressure.

2.1 Closure relations

The set of the governing equations are completed by including the closure term which are added to the source term in the momentum equations. The liquid pressure correction term is represented as; $P_c \frac{\partial \alpha_L}{\partial x} = \alpha_L \rho_L g \frac{\partial h_L}{\partial x} \cos\theta$, while the gas pressure correction term is neglected i. e $\Delta P_{Gi} = 0$ (Drew & Passman, 1999) and kwnown as the single pressure model.

The wall shear stress and interfacial shear stress for both phases k (G-gas and L-liquid) given as (Bonizzi, 2001);

$\tau_k = \frac{1}{2} f_k \rho_k V_k |V_k|$; $\tau_I = \frac{1}{2} f_I \rho_G (V_G - V_L)|V_G - V_L|$.

2.2 Method of Solution

The equations in (1)-(4) can be rewritten to a more compacted matrix form as;

$$\frac{\partial Q}{\partial t} + \frac{\partial E(Q)}{\partial x} = H \frac{\partial Q}{\partial x} + S \quad (5)$$

where Q is a vector of unknowns, F is a physical flux vectors, H contains non-conservative terms that exit in the model and S is a vector of algebraic source terms given respectively as follows:

$$Q = \begin{bmatrix} \rho_G \alpha_G \\ \rho_L \alpha_L \\ \rho_G \alpha_G V_G \\ \rho_L \alpha_L V_L \end{bmatrix}, \quad E = \begin{bmatrix} \rho_G \alpha_G V_G \\ \rho_L \alpha_L V_L \\ \rho_G \alpha_G V_G^2 + \alpha_G P \\ \rho_L \alpha_L V_L^2 + \alpha_L P \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & P/\rho_L & 0 & 0 \\ 0 & \frac{(P - P_c)}{\rho_L} & 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 0 \\ 0 \\ I_{fG} - I_s + I_{Gw} \\ I_{fL} - I_s + I_{Lw} \end{bmatrix} \quad (6)$$

2.3 Implicit Steger-Warming Flux Vector Splitting Scheme

The governing equations for the one dimensional single pressure two-fluid model are solved numerically which require numerical schemes by the finite difference method of discretization (Anderson, 1995; Hoffman & Chaing, 2000). The implicit Steger-Warming flux vector splitting method (FSM) is used as the numerical scheme.

The flux vector E and the flux Jacobian matrix A are splitted as $E = E^+ + E^-$ and $A = A^+ + A^-$ respectively.

A backward difference approximation is used for the positive terms and a forward difference approximation is used for the negative terms. Hence, considering the first-order approximations, the following finite difference equation is obtained;

$$\left[I+\frac{\Delta t}{\Delta x}\{(A^{+}{}_{i}+A^{+}_{i-1}+A^{-}_{i+1}-A^{-}_{i})\{-\Delta tB_{i}\}\}\right]\Delta Q$$

$$=-\Delta t\frac{1}{\Delta x}[(E^{+}_{i}-E^{+}_{i-1}+E^{-}_{i+1}-E^{-}_{i})+S_{i}] \qquad (7)$$

Rearranging the above equation in terms of the grid point i for the Jacobian matrix A (or say for the right hand side of equation (7) we get; $-\left(\frac{\Delta t}{\Delta x}A^{-}_{i+1}\right)\Delta Q_{i-1}+\left[I+\frac{\Delta t}{\Delta x}(A^{+}{}_{i}-A^{-}_{i})-\Delta tB_{i}\right]\Delta Q_{i}$

$$+\left(\frac{\Delta t}{\Delta x}A^{+}_{i+1}\right)\Delta Q_{i+1}=-\frac{\Delta t}{\Delta x}\left(E^{+}_{i}-E^{+}_{i-1}+E^{-}_{i+1}-E^{-}_{i}\right)+\Delta tS_{i} \qquad (8)$$

The linearized equation of (8) is expressed as;

$$-\left(\frac{\Delta t}{\Delta x}A^{0}_{i-1}\right)Q^{n+1}_{i-1}+\left(1+\frac{\Delta t}{\Delta x}(A^{0+}_{i}-A^{0-}_{i})-\Delta tB^{0}_{i}Q^{n+1}_{i}\right)+\left(\frac{\Delta t}{\Delta x}A^{0-}_{i+1}\right)Q^{n+1}_{i+1}=Q^{n}_{i} \qquad (9)$$

3.0 Results and Discussion

The numerical scheme of Steger Warming flux splitting method described earlier has been applied to the two-fluid model to analyse the numerical effects for various mesh refinements that is discretization of cells and time evolutions for two test cases such as the water faucet problem and phase separation problem.

3.1 Phase Separation test case

The phase separation flow test case reported by Paillere, Corre [32] and Städtke, Franchello (1997), a mixture consisting of air and water separated under the action of gravity. A vertical tube of length $L=7.5m$ is considered, with the coordinate system taking such that $x=0$ corresponds to the top of the vertical tube, and $x=7.5$ corresponds to the bottom of the tube. The boundary conditions with both ends of the tube closed that is $V_l 0=V_g 0=0$ m/s are applied. The pressure at the initial is $P=1.10^5$ Pa and has a liquid volume of fraction $\alpha_l=0.5$. At $t=0s$, the phases begin to separate due to the influence of gravity, which is considered in the source term. An approximate analytical solution for the liquid fraction and velocity was formed assuming that the liquid accelerated by the influence of gravity only, until it is abruptly brought into stagnation at the lower part of the tube (Evje & Flattern, 2003). A mesh of 100 cells is used, and the calculation is carried out until steady state is reached. The phases should be fully separated in the idealized case. The phase separation test case is shown in Fig. 11.

The liquid fraction and liquid velocity for time at $t=0.6s$ are compared to the analytical solution in order to check the accuracy and convergence of the numerical scheme and are presented in Figure 3 &4.

3.2 Time Evolution

The time evolution for the liquid fraction, void fraction and liquid velocity are presented in Figure 5-8. The time variation shows that the two volume fraction fronts at the top and bottom of the pipe. These fronts meet slowly and a stationary state is then formed, both phases are fully separated. It is observe that a steady state occurs after 0.8s, and then later part the volume of fractions results are thereby overlapping.

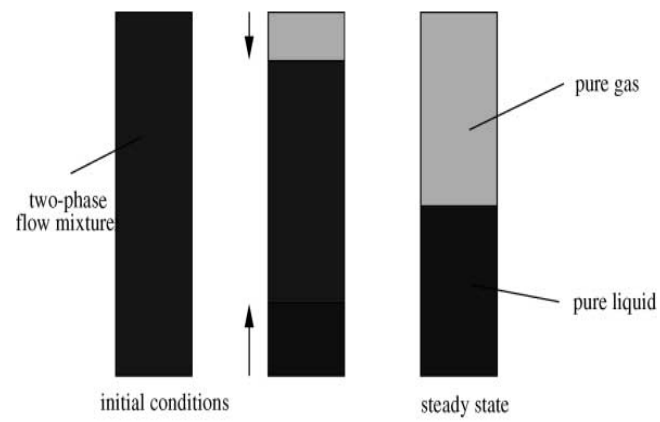

Figure 2 The phase separation test case

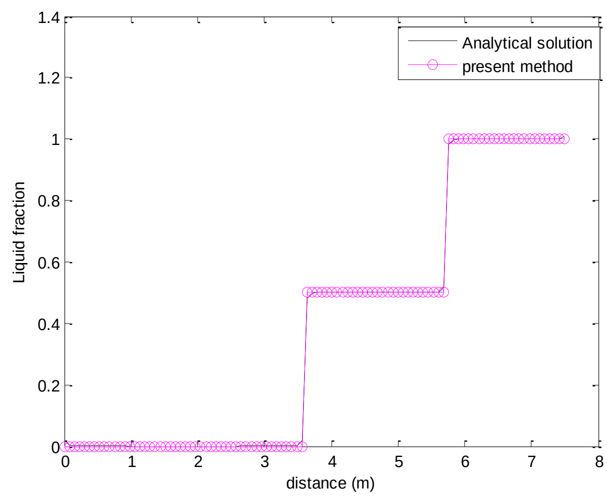

Figure 3 Analytical solution for liquid fraction at t=0.6 s and 100 cells Steger Warming flux vector splitting scheme

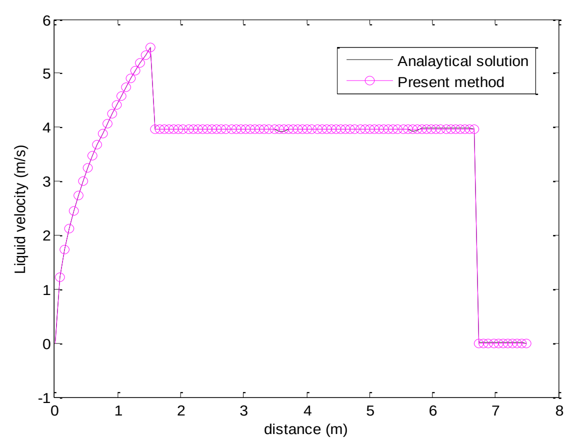

Figure 4 Analytical solution for liquid velocity at t=0.6 s and 100 cells compared to the Steger Warming flux vector splitting scheme

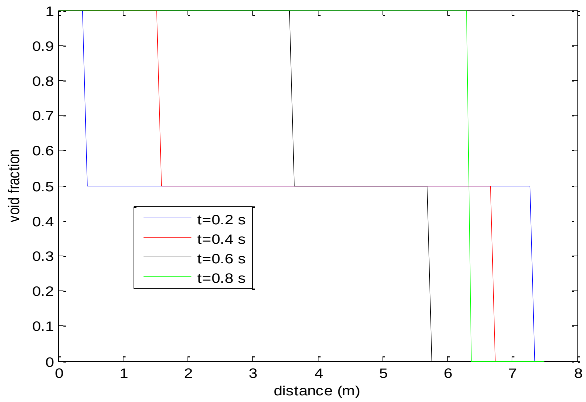

Figure 5 Void fraction with a function time evolution using Steger Warming flux vector splitting scheme.

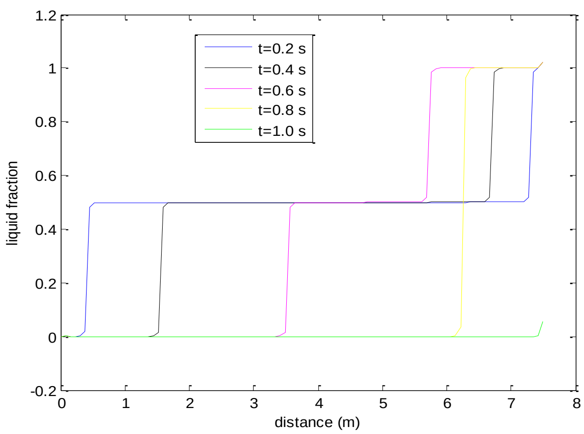

Figure 6 Time evolution for the liquid fraction using Steger Warming flux vector splitting scheme

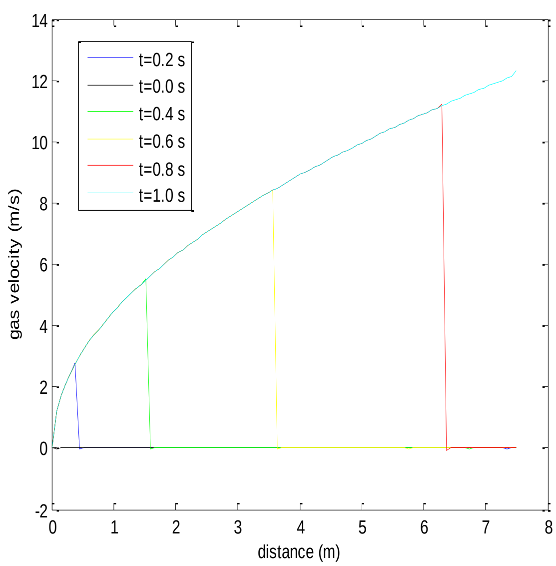

Figure 7 Time evolution for the gas velocity for separation case problem Steger Warming flux vector splitting scheme

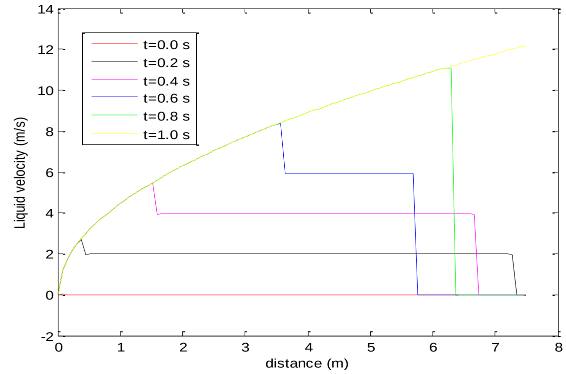

Figure 8. Time evolution for the liquid velocity for separation case problem using Steger Warming flux vector splitting scheme.

3.3 **Mesh Refinement**

Figure 9-12 shows the results for different discretization sizes with 25, 50, 100 and 200 cells for the liquid fraction, liquid velocity and gas velocity at time $t = 0.6s$ using Steger Warming flux splitting scheme.

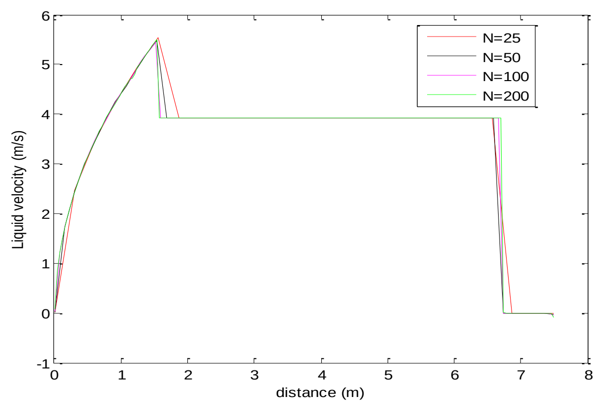

Figure 9 Liquid velocity for different cells at t=0.6s using Steger Warming flux vector Splitting Scheme.

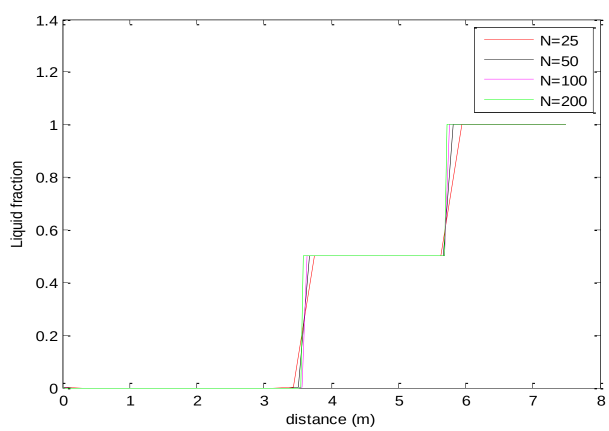

Figure 10 Mesh refinement for the liquid fraction at time t=0.6s using Steger Warming flux vector splitting scheme

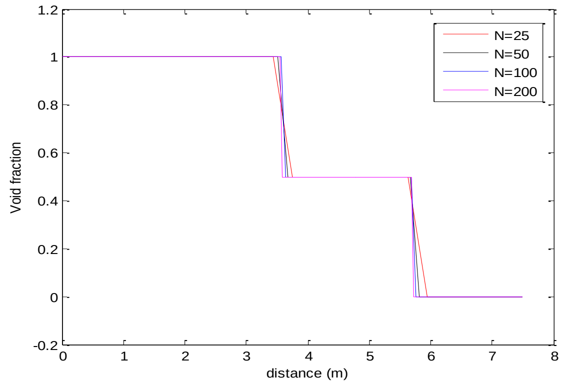

Figure 11 Mesh refinement for the void fraction at time t=0.6s for separation case problem using Steger Warming flux vector splitting scheme

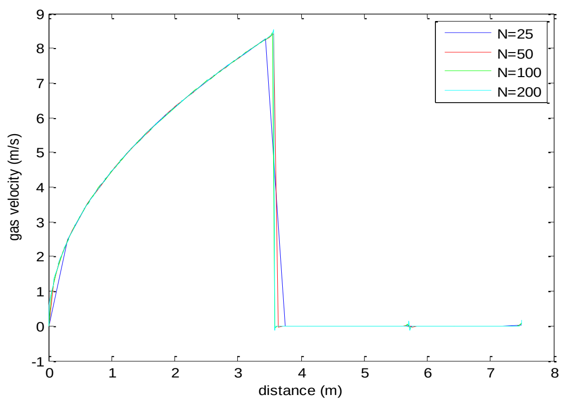

Figure 12 Mesh refinement for the gas velocity at time t=0.6s for separation case problem using Steger Warming flux vector splitting scheme.

5.0 Conclusions

A one-dimensional single pressure two-fluid model was considered that described mathematically a system of partial differential equations of two-phase flow for predicting and modelling the flow physics of two-phase gas-liquid flow in a pipe. The model is obtained by integrating (area averaging) the flow properties over the cross-sectional area of the flow. The dynamic interaction between the phases across the interfaces is modeled using inter-phase forces that appear as source terms in the transport equations. This makes the model capable of predicting the flow physics in two-phase flow systems. The transient one dimensional two fluid equations were solved numerically by the implicit Steger-Warming flux vector splitting technique. Numerical results on air-water compressible flow problems are preformed and analysed. A numerical computation for test case problem of separation was performed. The results for the comparison of the numerical scheme to the analytical solution was presented liquid fraction and likewise liquid velocity and void fraction for the separation case was also presented and shows good agreement with analytical solutions. The discretization for various cells and time evolution are presented that give an insight for the convergence and stability of the numerical scheme on the test case.

Declaration of conflict of interest

The manuscript titled "An Implicit Steger-Warming Flux Vector Splitting Method for the Transient Two-Phase Flow in a Pipe" by the authors, Yale I. D, and Norsarahaida Amin. This manuscript is a research article carried out during a study. The manuscript is an original work written by the authors and no part of it has been published before, nor is any part of it under consideration for publication at another journal and there is no conflict of interest among authors.

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